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Conservation laws with discontinuous flux: a short introduction

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Abstract Conservation laws with discontinuous flux have attracted recent attention both due to their numerous applications and the intriguing theoretical challenges raised by their well-posedness and numerical analysis. This introductory note states the basic problem considered in the eight contributions of this Special Issue. Three different types of applications are surveyed where these equations appear, motivated by spatially heterogeneous physical models, adjoint problems for parameter identification, and numerical methods for systems of conservation laws, respectively. Basic problems arising in the analysis of these equations are discussed, and the contributions of the Special Issue are presented.

Keywords Conservation laws · Discontinuous flux · Numerical methods · Transport equations · Well-posedness analysis

1 Introduction

In numerous models arising in engineering applications and the applied sciences, the one-dimensional flow of a quantity can be described by a *conservation law*, that is, a partial differential equation of the type

$$\partial_t \mathbf{u} + \partial_x \mathbf{f}(x, \mathbf{u}) = 0, \tag{1.1}$$

where $\mathbf{u}$ is a vector of state variables (for example, density, linear momentum and energy of a single fluid; or concentrations of individual species in a multi-species flow model), and we seek $\mathbf{u}$ as a function of position x and time t. The evolution of $\mathbf{u}$ is determined by the flux-density vector $\mathbf{f}$, which is a given function for each specific model under consideration. In most circumstances, $\mathbf{f}$ will depend nonlinearly on $\mathbf{u}$, and it is well known that the solution of (1.1) develops discontinuities, even if one starts from a smooth initial datum. We limit the further discussion to the scalar case, for which there is a closed well-posedness (existence and uniqueness) theory for so-called entropy solutions,

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that is, for discontinuous solutions of (1.1) that satisfy an admissibility condition. These results are, however, valid only in the case that $\mathbf{f}$ depends *smoothly* on x. For details, we refer to monographs on conservation laws, for example [1–4].

Conservation laws *with discontinuous flux* could formally still be written as (1.1), but their flux depends *discontinuously* on the spatial position x. These equations are a topic of intense current research, since they arise in a number of recent applications in engineering mathematics, but also due to their theoretical interest. They include, in particular, the case in which we switch between two different u-dependent fluxes across a fixed spatial position. In the scalar case, the prototype situation is the problem where we seek a solution u of

$$\begin{aligned} &\partial_t u + \partial_x g(u) = 0 \quad \text{for } x < 0, \\ &\partial_t u + \partial_x f(u) = 0 \quad \text{for } x > 0, \end{aligned} \tag{1.2}$$

where f and g are, say, smooth functions. This can compactly be rewritten as a conservation law

$$\partial_t u + \partial_x \mathcal{F}\big(\gamma(x), u\big) = 0, \tag{1.3}$$

where $\gamma(x)$ is the discontinuous parameter that "switches" between the fluxes $g(u)$ and $f(u)$, i.e.,

$$\mathcal{F}\left(\gamma(x), u\right) = \gamma(x) f(u) + (1 - \gamma(x))\, g(u), \quad \gamma(x) := \begin{cases} 0 & \text{for } x < 0, \\ 1 & \text{for } x \geq 0. \end{cases} \tag{1.4}$$

A particularly transparent example is the extension of the Lighthill–Whitham–Richards (LWR) [5,6] kinematic traffic model to roads with a discontinuously varying road surface, a scenario that was first studied by Mochon [7]. To explain it, consider first a one-dimensional, unidirectional and unbounded highway, and assume that $u(x,t)$ is the normalized density of cars, where $u = 0$ and $u = 1$ correspond to a free highway, on which cars travel at a given maximum speed $v_{\max}$, and to a bumper-to-bumper traffic jam, respectively. Now, assume that the velocity v of the car located at (x,t) is a function of $u(x,t)$ only, i.e., drivers adapt their velocity only to the density seen "on the spot", and that this velocity $v = v(u)$ is given by $v(u) = v_{\max}(1-u)$, where $1-u$ is a hindrance factor taking into account the presence of other cars. The one-dimensional equation of conservation of cars leads to the conservation law $\partial_t u + \partial_x \varphi(u) = 0$, where $\varphi(u) = uv(u) = v_{\max} u(1-u)$. Now, suppose that the segments of the road corresponding to $x < 0$ and $x > 0$ admit different maximum velocities $v_{\max}^{\text{left}}$ and $v_{\max}^{\text{right}}$, for example due to different pavements. This situation gives rise to a conservation law with discontinuous flux of the type (1.2), where

$$f(u) = v_{\max}^{\text{right}} u(1-u), \quad g(u) = v_{\max}^{\text{left}} u(1-u). \tag{1.5}$$

Suppose now that $u_-(t) = \lim_{x \uparrow 0} u(x,t)$ and $u_+(t) = \lim_{x \downarrow 0} u(x,t)$ denote the traces of u with respect to $x = 0$ at time t. Clearly, the fluxes to both sides of $x = 0$ must be equal at any time, which necessitates the Rankine–Hugoniot condition

$$f\left(u_+(t)\right) = g\left(u_-(t)\right). \tag{1.6}$$

The basic difficulty occurring in the well-posedness and numerical analysis for initial-value problems of (1.2) can be traced back to the fact that for a given value of, say, u_-, a corresponding trace of the solution u_+ at the other side of $x = 0$ is not uniquely determined by (1.6), since (as in our example) the fluxes $f(u)$ and $g(u)$ are not monotone in many cases of interest, and do not have the same range, so there may be several solutions, or no solution at all. The former case calls for an entropy condition, or admissibility condition, which is imposed (in addition to the Rankine–Hugoniot condition) on jumps of the solution across $x = 0$ while, in the latter, the solution cannot be a simple stationary jump from u_+ to another constant value; rather, we have to insert a sequence of jumps or rarefaction waves that separate u_+ from u_- and include $x = 0$; which, in turn, again calls for the application of an entropy condition. These difficulties are intimately related to the fact that a well-posedness theory for (1.1) with a function f that depends discontinuously on x cannot be formulated as the "limit case" of flux functions that depend smoothly on x. These problems evidently arise in the traffic model, but are typical for many other applications including sedimentation, flow in heterogeneous media, and ion etching. It turns out that the physics of each of these models singles out different desirable solutions, depending on, for example, whether or not we allow characteristic information to emerge to both sides of the location of the flux discontinuity. (For continuous sedimentation, for example, we do; for two-phase flow through heterogeneous porous media, we do not; the difference in behaviour of admissible solutions for both cases is sharply worked out in [8].)

Coming back to the traffic model, we mention that for the concave fluxes $f(u)$ and $g(u)$ given by (1.5), several entropy conditions, with various degrees of closeness to the physics of the traffic model, for jumps across $x = 0$ coincide, including minimality of the jump in density across $x = 0$; maximization of the flux through $x = 0$; the driver's ride impulse [9], stating that drivers tend to smooth out a sudden decrease in density; and the speedup impulse, which postulates that drivers always try to speed up, if possible, when approaching the interface [10].

Some applications require the use of several independent discontinuous parameters, which are collected in a vector $\boldsymbol{\gamma}(x)$, giving rise to equations of the type

$$\partial_t u + \partial_x \mathcal{F}(\boldsymbol{\gamma}(x), u) = 0; \tag{1.7}$$

others involve even another discontinuous coefficient, $\tilde{\gamma}(x)$, which controls the effectiveness of a (possibly degenerate) diffusion term:

$$\partial_t u + \partial_x \mathcal{F}(\boldsymbol{\gamma}(x), u) = \partial_x (\tilde{\gamma}(x)\partial_x A(u)). \tag{1.8}$$

Finally, one contribution in this collection involves an equation that is similar to, but is not a special case of (1.3), namely the linear transport equation

$$\partial_t u + a(x,t)\partial_x u = 0, \tag{1.9}$$

where the transport coefficient $a(x, t)$ is a known discontinuous function.

More generally speaking, there are three groups of current applications of conservation laws with discontinuous flux of the type (1.3) and related equations. The first group emerges from situations in which the flux discontinuity emerges from a spatially heterogeneous physical reality, and where the flux discontinuity is explicitly modeled. Models of this kind include traffic flow with heterogeneous surface conditions (as discussed above) and continuous sedimentation, which play a key role in several contributions of this issue, and will not be further outlined here. Furthermore, we mention models of two-phase flow in heterogeneous porous media [11,12], a population balance model of ball wear in grinding mills [13], and a model of endo-vascular treatment of abdominal aortic aneurysm [14] which are similar to the two previous models in that the discontinuity with respect to the spatial variable occurs at a fixed position. Another interesting model is the Liouville equation in classical mechanics with discontinuous potentials [15]. This equation is a particular linear transport equation with singular coefficients.

Models that are more general in the sense that the location of the flux discontinuity is not fixed a priori, but behaves as a free boundary, include a model of ion etching [16,17]. On the other hand, scalar conservation laws with discontinuous flux arise from the differentiation of Hamilton–Jacobi equations modelling shape-from-shading problems [18,19]. A thorough understanding of the problem (1.2), (1.3) is at the core of the well-posedness and numerical analysis of models of "network" type, in which the topology of the computational domain consists of a number of one-dimensional edges that are coupled at their endpoints forming "junctions" [20–22].

A second group of applications emerges from the study of inverse problems of a standard scalar conservation law having a continuous flux. One approach to identify the (unknown) flux f of a conservation law $\partial_t u + \partial_x f(u) = 0$ from observed solution data consists in the solution of a backwards-posed adjoint problem, in which the evaluation of a given solution of the direct problem acts as a transport coefficient. Since this solution will, in general, be discontinuous, we here obtain a problem of the type (1.9); the difficulty is that the term $a(x, t)\partial_x u$ in (1.9) is a product between two distributions, and is not well-defined a priori. A standard reference to a treatment that overcomes this problem is the paper by Bouchut and James [23]. The parameter identification problem has been studied in the context of a number of engineering applications, most notably, chromatography (see the contribution by James and Postel), but also for sedimentation and related models [24]. It is well known that the parameter-identification problem is in general ill-posed, which means that it is in general not possible to uniquely identify the flux from an "observed" solution. However, a partial analysis can be made to prove convergence of the adjoint scheme [25].

Finally, a more recent trend consists in the re-formulation of conservation laws with source terms (balance laws), and triangular systems of conservation laws, in terms of conservation laws with discontinuous flux, and to use local solvers for them as a building block of a numerical scheme. This means that the flux discontinuity does not arise from a spatially heterogeneous physical model, but is produced, for example, by the sequential cell-average discretization of variables in systems of conservation laws, where the discretized version of the equation for the n-th variable involves discrete values of the first $n-1$ variables as discontinuous parameters.

When the coefficients in the governing equations are discontinuous functions of the spatial variable, the "standard" well-posedness theory (in terms of existence, uniqueness, and stability of solutions) no longer applies and one needs to develop a new theory. Besides, one must construct new numerical methods and invent new analytical tools for analyzing these methods. A basic and well-known difficulty concerns the choice of an appropriate entropy condition to ensure that a relevant (weak) solution can be determined uniquely once the initial and boundary data have been prescribed. Another difficulty is that the discontinuities in the flux induce "resonance" [26], which implies a lack of a priori bounds on the spatial total variation of the solution, in marked contrast to the smooth x-dependent situation where the classical Kružkov-type theory applies. For example, when the initial data is approximated by a sequence of piecewise constant functions, this can cause the spatial variation to blow up as the discretization parameter tends to zero.

Over the past decade many authors have worked on conservation laws and other related equations with discontinuous flux, see for example [19,27–37]. Much effort has gone into the development of suitable notions of entropy solutions for which one can prove existence and uniqueness as well as convergence of numerical methods (such as front-tracking, upwind difference schemes, relaxation schemes, vanishing viscosity/smoothing of coefficients, etc.), at least for certain classes of equations. With no spatial total-variation bound available for the conserved quantity, sophisticated methods for establishing compactness have been employed to prove convergence of the approximate solutions and thereby providing existence results, including the singular mapping, compensated compactness, and kinetic methods. Some of these topics are touched upon for example in the contribution by Bürger, García, Karlsen, and Towers. Other numerical techniques that have been advanced for solving systems of conservation laws with discontinuous flux, and which are mainly justified by numerical experiments, include the so-called δ-mapping approach [38,39], which is a particular way of "pre-processing" solution data in each time step, such that a standard numerical scheme applied to the δ-mapped data approximates solutions of the discontinuous flux problem; and a characteristics-based relaxation method [40].

This Special Issue includes papers that were presented in a minisympsosium on "Conservation Laws and Related Equations with Discontinuous Flux: Theory, Numerics, Applications", which the guest editors had organized within the conference "The Mathematics of Finite Elements and Applications 2006 (MAFELAP 2006)", which took place at Brunel University, Uxbridge, UK, June 13–16, 2006, and some invited contributions. We would like to thank the organizers of MAFELAP 2006, in particular Professors John R. Whiteman and Norbert Heuer, for the possibility to organize this minisymposium, and we are grateful to the Editor-in-Chief of *Journal of Engineering Mathematics*, Professor Henk K. Kuiken, for offering the opportunity to publish this Special Issue, and his constant advice and support.

2 Papers of the Special Issue

The Special Issue starts with the paper "*Operating charts for continuous sedimentation IV: limitations for control of dynamic behaviour*" by Stefan Diehl. This paper concludes a series of papers by the same author, in which he explicitly constructs solutions to the clarifier–thickener model, which is one of the traditional models giving rise to a conservation law with discontinuous flux. In fact, this model features three different flux discontinuities, one of which corresponds to a singular source term. Despite its relative simplicity, this model produces a fairly involved solution behaviour. To make the model amenable to engineering applications, the author has developed the concept of "operating charts", which permit to "read off" from the flux plot predictions of the dynamic behaviour of the clarifier–thickener, and to take control actions to ensure stable operation. The present paper deals with the limitations to which control actions are subject.

While the first paper is concerned with limitations for the manual control of clarifier–thickeners, the second paper "*A regulator for continuous sedimentation in ideal clarifier–thickener units*" by the same author deals with a refined strategy for automatic clarifier–thickener control, which ensures that the sludge-blanket level (in other applications, called sediment level) remains near a reference value, or alternatively, the underflow concentration is larger or equal to a minimum value. The regulator is based on continuous measurement of the total mass of solids contained within the unit, which is achieved by balancing the solids feed and underflow. The advantage of this approach is that it is not necessary to measure the sludge-blanket level within the unit, which may be technically difficult.

The first two papers address a concrete engineering application whose main model ingredient is a conservation law with discontinuous flux, and which is typical of the first group of applications mentioned in the introduction. The third paper, "*Numerical gradient methods for flux identification in a system of conservation laws*" by François James and Marie Postel equally deals with an engineering application, namely the problem of flux identification (more precisely, identification of isotherms) in chromatography. The fluxes appearing in the model are not discontinuous, but one step of the parameter identification procedure consists in solving an adjoint problem, where a discontinuous solution of the direct problem appears in the coefficients of a linear, backward-in-time transport equation. Clearly, this problem represents the second group of applications.

The next paper in this collection, "*Entropy formulations for a class of scalar conservation laws with space-discontinuous flux functions in a bounded domain*" by Julien Jimenez and Laurent Lévi, considers again a problem that has a discontinuous flux as a model ingredient; this time, however, the setting is more abstract and the analysis can be applied to several physical models. More precisely, they consider a multi-dimensional mixed hyperbolic–parabolic problem posed on an underlying domain consisting of two parts, one part on which a hyperbolic equation lives and one complementary part on which a weakly degenerate parabolic equation lives. This setup is relevant in many applications in which the two domains would reflect different geological properties. This problem may also be viewed as a strongly degenerate parabolic equation with discontinuous coefficients. The crucial device to obtain a well-posed problem is to impose suitable conditions across the interface between the two regions. The paper by Levi and Jimenez successfully identifies such interface conditions, and establishes the existence and uniqueness of suitably defined entropy solutions satisfying these conditions for the homogeneous Dirichlet problem.

The next two contributions, "*Semi-Godunov schemes for general triangular systems of conservation laws*" and "*A large time-stepping scheme for balance equations*", both by Kenneth H. Karlsen, Siddharta Mishra and Nils Henrik Risebro, illustrate the value of solutions of conservation laws with discontinuous flux for the design of numerical schemes for certain systems of conservation laws with a flux that depends continuously on the spatial location. In the case of triangular systems, one has to deal with a scalar equation of the type $\partial_t u_1 + \partial_x f_1(u_1) = 0$ for a scalar quantity u_1, a second equation $\partial_t u_2 + \partial_x f_2(u_1, u_2) = 0$, and so on. In each time step, we may solve the first equation for u_1, insert the result into the second equation, solve for u_2, and so on. In this procedure, the updated solution of, say, u_1, will be in general discontinuous, and appears as a discontinuous parameter in the second equation, and so on. Numerical schemes for triangular systems that explicitly exploit this observation were termed *semi-Godunov schemes*. Their advantage is the fact that in each time step, locally only a hierarchy of scalar problems is solved, but a characteristic decomposition or solutions of full-size Riemann problems are not needed. The approach works for general triangular systems, but the authors focus on the specific engineering application of three-phase flow in porous media.

In the second paper, the authors study scalar balance equations of the type $\partial_t u + \partial_x f(u) = A(x, u)$, and construct numerical schemes that on one hand, preserve certain steady-state solutions of this equation, and on the other hand, converge rapidly to these solutions. The key idea is to subtract the primitive (with respect to x) of $A(x, u)$ from the flux $f(u)$, which leads to a homogeneous conservation law; and then, in time-stepping procedure, to replace the corresponding u-evaluation within that primitive by a numerical solution from the previous time step, which again leads to a scalar conservation law with discontinuous flux. Here, the treatment is general, but specific models giving rise to balance equations include, for example, the shallow water equations.

The contribution "*Fully adaptive multi-resolution schemes for strongly degenerate parabolic equations with discontinuous flux*" by Raimund Bürger, Ricardo Ruiz, Kai Schneider, and Mauricio Sepúlveda, presents an accurate and efficient numerical scheme for computing approximate entropy solutions (in the sense of [34]) strongly degenerate parabolic equations with discontinuous flux. To achieve high accuracy in discontinuous parts of the solution of such equations with a difference scheme defined on a uniform grid, it becomes necessary to employ a very fine spatial discretization parameter, which results in very time-consuming computations. The contribution of Bürger, Ruiz, Schneider, and Sepúlveda addresses this hurdle by augmenting a first order upwind (Engquist–Osher) scheme [34] with an adaptive multi-resolution algorithm. The purpose of a multi-resolution (wavelet) representation of the solution is to detect discontinuities and thus to allow for a locally refined grid by cutting off the irrelevant wavelet coefficients, thereby reducing the number of flux evaluations in the upwind scheme. The authors present several numerical examples related to traffic flow with driver reaction and a clarifier–thickener model. These examples illustrate the added value of the multi-resolution algorithm.

The final contribution of the Special Issue is "*A family of schemes for kinematic flows with discontinuous flux*" by Raimund Bürger, Antonio García, Kenneth H. Karlsen, and John D. Towers. Kinematic models are relevant in a number of applications such as multi-phase flows (sedimentation of polydisperse suspensions, separation of oil-in-water dispersions, etc.) and traffic flow of vehicles on a highway. Conservation equations for the concentration $\phi_i = \phi(x, t)$, $i = 1, \dots, N$, of each species can be written down effortlessly, and the models appropriate to the matter at hand become systems of conservation laws for which the flux function of the ith equation is $f_i(x, \phi_1, \dots, \phi_N) = \phi_i v_i$; the ith velocity $v_i = v_i(x, \phi_1, \dots, \phi_N)$ is an explicitly given function of the vector of concentrations of all species, which may also depend discontinuously on the spatial variable x. The purpose of the contribution of Bürger, García, Karlsen, and Towers is to exploit the specific "concentration times velocity" structure of these kinematic fluxes to formulate and partly analyze very simple difference schemes. Several numerical experiments are presented to show that these schemes compare well with more complicated schemes, but they are simpler to implement as no approximate Riemann solver is involved; calculation of eigenvalues, eigenvectors, field-by-field decompositions, flux vector splittings, etc., are not needed. The convergence analysis, which applies to the scalar case, brings with it a compactness proof involving a novel uniform but local estimate of the spatial total variation of the approximate solutions.

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